

## DISCUSSION: “A SIGNIFICANCE TEST FOR THE LASSO”

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We congratulate the authors for an interesting article and an innovative proposal to testing the significance of the predictor variables selected by the Lasso. There is much material for thought and exploration. Research on high-dimensional regression has been very active in recent years, but most of the efforts have so far focused on estimation. Despite the popularity of the Lasso as a variable selection technique, the problem of making valid inference for a model chosen by the Lasso is largely unsettled. The current paper pinpoints some of the challenges in making valid inference in the high-dimensional setting and presents a thought-provoking approach to address them.

Following the notation used in the paper, let  $A$  be the model selected at the  $k$ th step of either the Lasso or forward stepwise regression and  $j$  be the index of the variable to be added in the next step. This paper considers the problem of testing the null hypothesis that the underlying model corresponding to the true regression coefficient vector  $\beta^*$  is nested in the current selected model, that is,

$$H_0 : \text{supp}(\beta^*) \subseteq A.$$

As pointed out in the paper, a classical approach to testing two fixed nested models  $A$  and  $A \cup \{j\}$  is the chi-squared test, which is based on the test statistic

$$R_j = (\text{RSS}_A - \text{RSS}_{A \cup \{j\}}) / \sigma^2$$

and compares it to the quantile of the  $\chi_1^2$  distribution. The test fails, as noted, when applying to the forward stepwise regression or the Lasso in a vanilla fashion because it fails to account for the fact that neither  $A$  nor

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$\{j\}$  is fixed. The randomness of  $A$  can be addressed using a conditional argument as suggested by the authors. The effect of the way that the new index  $j$  is selected is more subtle. The seemingly lack of a remedy to this problem motivates the authors to focus on the Lasso and to propose the so-called covariance test statistic

$$(1) \quad \begin{aligned} T_k &= (\langle y, X\hat{\beta}(\lambda_{k+1}) \rangle - \langle y, X_A\tilde{\beta}_A(\lambda_{k+1}) \rangle) / \sigma^2 \\ &= R_j - \lambda_{k+1} (\langle s_{A \cup \{j\}}, \hat{\beta}_{A \cup \{j\}}^{\text{LS}} \rangle - \langle s_A, \hat{\beta}_A^{\text{LS}} \rangle) / \sigma^2, \end{aligned}$$

where  $s_A$  and  $s_{A \cup \{j\}}$  are, respectively, the vector of signs of the nonzero regression coefficients for the Lasso at the  $k$ th and  $(k+1)$ st steps, and  $\hat{\beta}_M^{\text{LS}} = (X_M^\top X_M)^{-1} X_M^\top y$  is the least squares estimate under model  $M$ . In effect, the second term on the right-hand side of (1) can be viewed as a correction factor to account for the fact that the next index  $j$  is not fixed, but selected through the penalized  $\ell_1$  minimization. It is shown in the present paper that under  $H_0$ , the limiting null distribution of  $T_k$  is either  $\text{Exp}(1)$  or stochastically smaller than  $\text{Exp}(1)$ , and the paper proposed a test for the null hypothesis  $H_0$  based on this fact.

In this discussion, we introduce and explore a perhaps simpler and more generic correction factor whose simplicity makes it an appealing alternative to the current proposal. Furthermore, it can be easily extended to other settings such as logistic regression and Cox proportional hazards regression.

*An alternative test.* Our proposal is based on the observation that for a given subset  $A$ , the next selected index  $j$  is not an arbitrary index in  $A^c$ . It is instructive to first look at the case of orthogonal design where it is clear that for both forward stepwise regression and the Lasso,  $j$  can be identified with

$$R_j = \max_{m \in A^c} R_m.$$

As a result, although for a fixed index  $m \in A^c$ ,  $R_m$  is a  $\chi_1^2$  distributed random variable,  $R_j$ , which is the maximum of  $R_m$  for all  $m \in A^c$ , is not  $\chi_1^2$  distributed. Note that, conditioning on the design matrix  $X$ ,  $R_m$ 's are independent  $\chi_1^2$  random variables. Therefore, the conditional distribution of  $R_j$  given  $X$  can be easily deduced from the distribution of the maxima of independent Gaussian random variables [see, e.g., de Haan and Ferreira (2006)]. In particular, in a high-dimensional setting where  $p$  is large and  $|A|$  is relatively small, the null distribution of  $R_j$  can be well approximated by a Gumbel distribution (of type I). More specifically, it can be shown that

$$(2) \quad R_j - 2\log(|A^c|) + \log \log(|A^c|) \xrightarrow{d} \text{Gumbel}(-\log \pi, 2) \quad \text{as } p \rightarrow \infty,$$

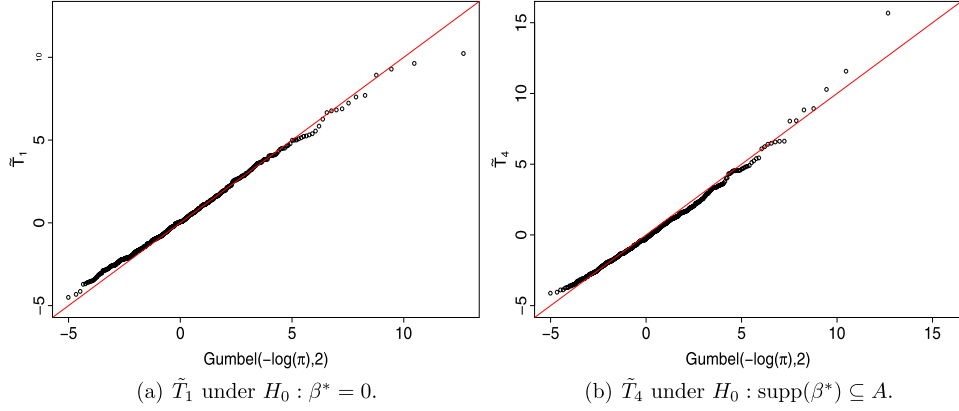


FIG. 1. Comparisons of the empirical distributions with the reference distribution for  $\tilde{T}_k$  under the orthogonal design.

where the distribution function of a random variable  $G$  following  $\text{Gumbel}(-\log \pi, 2)$  is given by

$$\mathbb{P}(G \leq x) = \exp(-\exp(-(x + \log \pi)/2)).$$

This motivates us to consider the following test statistic:

$$(3) \quad \tilde{T}_k = R_j - 2 \log(|A^c|) + \log \log(|A^c|)$$

and compare  $\tilde{T}_k$  with the quantile of  $\text{Gumbel}(-\log \pi, 2)$  distribution for testing the null hypothesis  $H_0$ . More specifically, for any given  $0 < \alpha < 1$ , we will reject  $H_0$  at the  $\alpha$  level if and only if  $\tilde{T}_k \geq q_{1-\alpha}^G$  where  $q_{1-\alpha}^G$  is the  $1 - \alpha$  quantile of  $\text{Gumbel}(-\log \pi, 2)$ .

To illustrate the accuracy of the reference distribution, we first repeated the experiment considered in the paper with  $n = 100$  observations and  $p = 50$  variables under the orthogonal design. When the true model is  $\beta^* = 0$  and, therefore, the null hypothesis holds, we computed  $\tilde{T}_1$  for 500 simulated datasets. The Q-Q plot of the observed  $\tilde{T}_1$  versus its reference distribution  $\text{Gumbel}(-\log \pi, 2)$  is given in the left panel of Figure 1. Similarly, the right panel of Figure 1 gives the Q-Q plot for  $\tilde{T}_4$ , again computed from 500 simulated datasets, when  $\beta^* = (6, 6, 6, 0, \dots)^\top$ .

The strength of  $\tilde{T}_k$  comes from the robustness of its limiting distribution under correlated designs. When  $X^\top X \neq I$ ,  $R_m$ 's are no longer independent but they are still marginally  $\chi_1^2$  distributed random variables. The distribution of  $R_j = \max_{m \in A^c} R_m$  again can be deduced from that of the maxima of a Gaussian process. In particular, it can be shown that the limiting Gumbel distribution given by (2) continues to hold under fairly weak conditions on the dependence structure [see, e.g., Leadbetter, Lindgren and Rootzén (1983)]. To verify the accuracy of the Gumbel approximation under dependency, we repeated the previous example with  $\beta^* =$

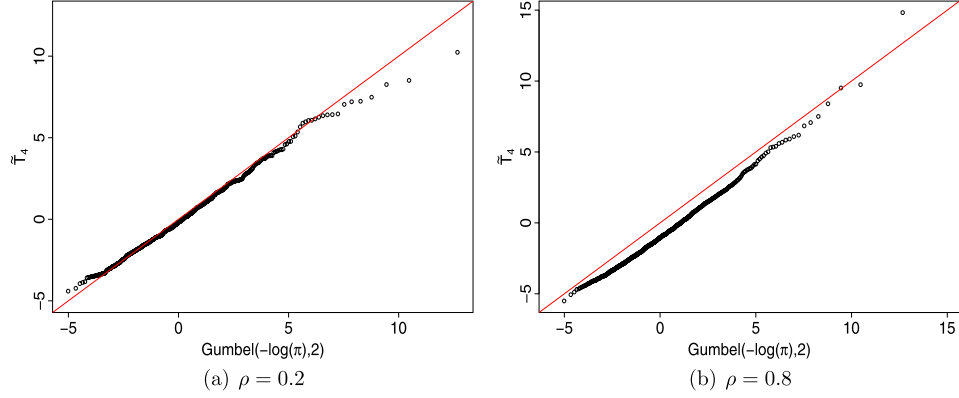


FIG. 2. Comparisons of the empirical distributions with the reference distribution for  $\tilde{T}_4$  under the null  $H_0: \text{supp}(\beta^*) \subseteq A$  with the AR(1) design.

$(6, 6, 6, 0, \dots)^\top$ . But instead of the orthogonal design, the design matrix is now generated from a multivariate normal distribution with mean zero and covariances  $\text{cov}(X_i, X_j) = \rho^{|i-j|}$ . The left panel of Figure 2 corresponds to  $\rho = 0.2$  and right panel to  $\rho = 0.8$ , both suggesting that the limiting distribution  $\text{Gumbel}(-\log \pi, 2)$  continues to provide a reasonable approximation to the null distribution of  $\tilde{T}_k$ . In contrast, numerical results show that the distribution of  $T_k$  could deviate significantly from the reference distribution  $\text{Exp}(1)$  under the correlated designs, and thus comparing it to  $\text{Exp}(1)$  could be rather conservative in the correlated case.

*General nonlinear  $\ell_1$  regularization problems.* The advantages of the test statistic  $\tilde{T}_k$  proposed in (3) are in its simplicity and generality. The correction factor utilized by  $\tilde{T}_k$  depends only on the number of remaining variables, and is straightforward to evaluate. This makes it particularly appealing when considering extensions to more general nonlinear  $\ell_1$  regularization problems where the exact tuning parameter  $\lambda_{k+1}$  for the next knot is typically not known in closed form and often has to be approximated using an iterative procedure. On the other hand, the validity of the Gumbel distribution as the reference distribution under  $H_0$  remains when  $R_j$  is replaced by the commonly used likelihood ratio test statistics.

To illustrate this point, we consider a logistic regression model where the true regression parameter is  $\beta^* = 0$ . With  $n = 100$  observations on a binary response and  $p = 50$  covariates independently generated from the standard normal distribution. Same as before, the experiment was repeated for 500 times; the Q-Q plot of the resulting statistic  $\tilde{T}_1$  with respect to the  $\text{Gumbel}(-\log \pi, 2)$  distribution is given in the left panel of Figure 3. The right panel of Figure 3 shows the results from a similar experiment for

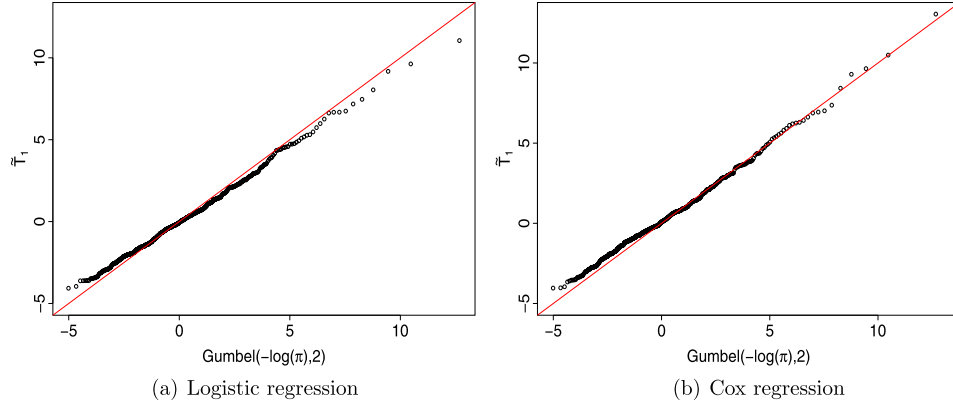


FIG. 3. Reference distribution for  $\tilde{T}_1$  under  $H_0: \beta^* = 0$  for logistic regression and Cox's proportional hazards model.

Cox proportional hazards regression where the response was generated from  $\text{Exp}(1)$  with 10% censoring. In both cases, the reference  $\text{Gumbel}(-\log \pi, 2)$  distribution provides a good approximation to the null distribution of the test statistic  $\tilde{T}_1$ .

*Summary.* The Lasso is a popular method for the high-dimensional linear regression and it is important to make statistical inference for a model chosen by the Lasso. The authors raise intriguing inferential questions in the paper and propose a novel method to addressing them. The work sheds new insight on high-dimensional model selection using the Lasso and will definitely stimulate new ideas in the future. The alternative test based on the test statistic  $\tilde{T}_k$  given in (3) merits further investigation for linear regression, logistic regression and Cox proportional hazards regression, under the high-dimensional setting. We thank the authors for their interesting work.

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